Noncanonical quantization of gravity. II. Constraints and the physical Hilbert space

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Abstract

The program of quantizing the gravitational field with the help of affine field variables is continued. For completeness, a review of the selection criteria that singles out the affine fields, the alternative treatment of constraints, and the choice of the initial (before imposition of the constraints) ultralocal representation of the field operators is initially presented. As analogous examples demonstrate, the introduction and enforcement of the gravitational constraints will cause sufficient changes in the operator representations so that all vestiges of the initial ultralocal field operator representation disappear. To achieve this introduction and enforcement of the constraints, a well characterized phase space functional integral representation for the reproducing kernel of a suitably regularized physical Hilbert space is developed and extensively analyzed.

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I. INTRODUCTION AND THE MAIN POINTS OF PAPER I

In a previous paper [1] (hereafter referred to as P-I) an introduction and outline of a program of noncanonical quantization of the gravitational field was presented. The key concepts in the present approach are (i) a careful selection of the basic kinematical variables, (ii) the use of a quantization procedure that treats all constraints alike, (iii) the use of ultralocal field operator representations prior to introducing constraints, and (iv) the imposition of the gravitational constraints, a process in which all traces of the temporary ultralocal representation characteristics are replaced with the physically relevant ones. In this section, we briefly review topics (i), (ii), and (iii) which have largely been discussed already in P-I.

A. Nature of the basic gravitational variables

One of the central requirements for the present program is the preservation, on quantization, of the positive-definite character of the spatial part of the classical metric $g_{ab}(x)$, $a,b\in\{1,2,3\}$ (or more generally $a,b\in\{1,\ldots,s\}$ in an s-dimensional space, $s\geq 1$). Insisting on this requirement leads us to adopt *affine commutation relations* (in contrast, for example, to canonical commutation relations) which are expressed in terms of the local metric field operators $\hat{g}_{ab}(x)$ $[=\hat{g}_{ba}(x)]$ (which were denoted by $\sigma_{ab}(x)$ in P-I) and suitable local "scale" field operators $\hat{\pi}^c_d(x)$ (which were denoted by $\kappa^c_d(x)$ in P-I). The word "local" here is intended to mean that these expressions only become operators after smearing with suitable spatial test functions. In units where $\hbar=1$, which are commonly assumed throughout this paper, the basic set of affine commutation relations reads

$$\begin{aligned}
[\hat{\pi}^a_b(x),\,\hat{\pi}^c_d(y)] &= \tfrac{1}{2}i\,[\delta^c_b\hat{\pi}^a_d(x)-\delta^a_d\hat{\pi}^c_b(x)]\,\delta(x,y)\;,\\
[\hat{g}_{ab}(x),\,\hat{\pi}^c_d(y)] &= \tfrac{1}{2}i\,[\delta^c_a\hat{g}_{bd}(x)+\delta^c_b\hat{g}_{ad}(x)]\,\delta(x,y)\;,\\
[\hat{g}_{ab}(x),\,\hat{g}_{cd}(y)] &= 0\;.
\end{aligned}\qquad(1)$$

These commutation relations are translations of identical Poisson brackets (modulo $i\hbar$, of course) for corresponding classical fields, namely, the spatial metric $g_{ab}(x)$ and the mixed-valence ("scale") field $\pi^c_d(x)\equiv g_{bd}(x)\pi^{bc}(x)$, along with the usual Poisson brackets between the metric field $g_{ab}(x)$ and

the canonical momentum field $\pi^{cd}(x)$. While *classically* there is essentially nothing to be gained by using the field $\pi^c_d(x)$ rather than $\pi^{cd}(x)$, *quantum mechanically* the situation changes completely. This change arises because the affine commutation relations—and *only* the affine commutation relations—admit local self-adjoint operator solutions for both $\hat{g}_{ab}(x)$ and $\hat{\pi}^c_d(x)$ which in addition have the property that $\hat{g}_{ab}(x) > 0$ for all x. This latter property means that for any nonvanishing set $\{u^a\}$ of real numbers and any nonvanishing, nonnegative test function, $f(x) \geq 0$, that

$$\int f(x)\, u^a\, \hat{g}_{ab}(x)\, u^b\, d^3x > 0 \; . \tag{2}$$

Other choices of basic variables fail this test. For example, self-adjoint canonical variables lead to metrics that have spectra unbounded below as well as above, while triad fields and their canonical partners lead to metrics that are nonnegative, but not necessarily positive definite.

B. How quantum constraints are to be imposed

Since gravity is a reparametrization invariant theory, it follows that the dynamics—indeed, *the entire physical content of the gravitational field*—enters through imposition of the relevant constraints, specifically, the diffeomorphism and Hamiltonian constraints [2, 3]. These constraints lead to an open set of *first-class* constraints in the classical theory, but on quantization and in virtue of an anomaly (or, alternatively, a factor ordering problem), they give rise to a set of operator constraints that, partially at least, are *second class* in nature. There exist several methods to deal with the quantum theory of second-class constraints in the literature, but generally such methods treat the first- and second-class constraints, and thereby the variables on which they depend, in fundamentally different ways.

An exception to the rule of a different operator treatment for first- and second-class constraints is offered by the *projection operator method* approach to the quantization of systems with constraints [4, 5, 6]. The projection operator method was already discussed in P-I, and that discussion also included several elementary applications of the technique to simple, few degree-of-freedom systems. We do not repeat that discussion here. Instead, we simply observe that, rather than impose the self-adjoint quantum constraints in the idealized (Dirac) form $\Phi_\alpha|\psi\rangle_{phys} = 0$, $\alpha \in \{1, \ldots, A\}$, on vectors $|\psi\rangle_{phys}$

in a putative physical Hilbert space, $|\psi\rangle_{phys} \in \mathfrak{H}_{phys}$, we define a (possibly regularized) $\mathfrak{H}_{phys} \equiv \mathbb{E}\mathfrak{H}$, in which $\mathbb{E}$ denotes a *projection operator* defined by

$$\mathbb{E} = \mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2) \ , \tag{3}$$

where $\delta(\hbar)$ is a positive *regularization parameter* (not a δ-function!) and we have assumed that $\Sigma_\alpha \Phi_\alpha^2$ is self adjoint. As a final step, the parameter $\delta(\hbar)$ is reduced as much as required, and, in particular, when some second-class constraints are involved, $\delta(\hbar)$ ultimately remains strictly positive. This general procedure treats all constraints simultaneously and treats them all on an equal basis.

C. Why ultralocal fields are relevant

In quantizing any theory with constraints, including reparametrization invariant theories, we invariably follow the rule: *Quantize first, reduce second*, also used by Dirac [3]. ⟦**Remark:** Some other quantization procedures reduce (i.e., impose constraints) first and quantize second, a scheme that may lead to different and generally incorrect results, especially when the physical phase space (the quotient of the constraint surface by the gauge transformations) is non-Euclidean [7]. We do not comment further on these alternative procedures.⟧ It must be strongly emphasized that in the initial quantization phase of this pair of operations, one must remain neutral toward, or even better, blind to any specifics of the constraints to be imposed. In particular, in the initial, quantization phase, reparametrization invariance requires that basic fields are statistically independent at any spatial separation; correlations between fields that are spatially separated, and which originate from the particular physics of the theory under discussion, will arise only after the relevant constraints are imposed. Before the constraints are ever discussed, the nature of the field operators is *ultralocal*, the name given to field-operator representations which are, in fact, statistically independent for all distinct spatial points. The primary representation already presented in P-I for the basic local affine quantum field operators, and before constraints have been introduced, is for this very reason ultralocal in character. In the present paper, we shall discuss the relevant constraints for gravity and argue, to the extent possible at present, that the imposition of the constraints leads

to field operator representations that are no longer ultralocal in character. Indeed, we shall argue that all traces of an initial ultralocal representation disappear when the constraints are fully enforced. [**Remark:** As an illustration of these general procedures, it is important at this point to observe that a relativistic free field of mass m (and, in addition, well-defined interacting fields) can be quantized starting from a reparametrization invariant formulation and with an initial field operator representation that is ultralocal in nature [8]. After imposing the appropriate constraint, it may be shown, by a suitable modification of the reproducing kernel—see Sec. IV for a brief discussion—how the conventional and *non*ultralocal field operator representation for the relativistic free field of an *arbitrary* mass m emerges, as well as how the conventional propagator for such a system also arises.]

D. Outline of remaining sections

In Sec. II, we discuss in some detail the formulation of the initial stage of quantization in the absence of the gravitational constraints. In Sec. III, the gravitational constraints are introduced, and a novel, but generally familiar, functional integral representation of the desired expressions is developed. Finally, in Sec. IV, and aided by the use of several analogies, we present a lengthy discussion of the virtues, as we see them, of the specific functional integral representation developed in this article.

II. REPRODUCING KERNEL FOR THE ORIGINAL HILBERT SPACE, AND ITS FUNCTIONAL INTEGRAL REPRESENTATION

A. Basic field operator representation and the original Hilbert space

For reasons briefly sketched above, we adopt as basic local field operators the positive-definite, local self-adjoint metric operators $\hat{g}_{ab}(x)$, and the local self-adjoint "scale" operators $\hat{\pi}^c_d(x)$, which satisfy the affine commutation relations (1) in the "original" Hilbert space $\mathfrak{H}$. Here $x = \{x^a\}^3_{a=1}$ denotes

the coordinates (each with the dimension of *length*) of a point in a classical topological space $\mathcal{S}$, which, for the sake of discussion, we may assume is topologically equivalent to $\mathbb{R}^3$. The term $\delta(x,y)$ which enters the affine commutation relations is a Dirac δ-function "scalar density" with dimensions $(length)^{-3}$. Moreover, the affine commutation relations uniquely tell us that $\hat{\pi}^c_d(x)$ transforms as a mixed-valence tensor density of weight one and has the engineering dimensions of an *action* (due to $\hbar$) times $(length)^{-3}$, i.e., M/LT in terms of *mass* (M), *length* (L), and *time* (T). We require that $\hat{g}_{ab}(x)$ transform as a covariant tensor of rank two, and that $\hat{g}_{ab}(x)$ is dimensionless.

An auxiliary—but temporary—structure

For purposes of the present section, we shall need to augment the classical topological space $\mathcal{S}$ with one additional structure, namely a volume form. Specifically, we adopt a real, positive scalar density of weight one, $b(x)$, $0 < b(x) < \infty$, $x \in \mathcal{S}$, with which we may define, at each point, an invariant volume form $dV \equiv b(x)\, d^3\!x$. Additionally, in keeping with its transformation properties, we further insist that $b(x)$ has the dimensions of L^{-3}.

The introduction of the auxiliary structure represented by $b(x)$ is **required** *before the constraints are introduced, but whatever choice is made, it will* **disappear completely** *after the constraints are fully enforced.*

More explicitly, several arguments are offered below to justify the introduction of the auxiliary structure represented by the function $b(x)$, $x \in \mathcal{S}$, prior to the introduction of the constraints. In Secs. III and IV, we will observe that we expect all the irrelevant freedom present in $b(x)$ to be "squeezed out" by the constraints and replaced with another, *non*arbitrary structure with the same dimensions. This is exactly the process observed in other cases that have been explicitly dealt with, and the particular case of a reparametrization invariant relativistic free field will be summarized in Sec. IV [8].

Original reproducing kernel

Dual to the local operators $\hat{g}_{ab}(x)$ and $\hat{\pi}^c_d(x)$, we next introduce two real, c-number functions $\pi^{ab}(x)$ $[=\pi^{ba}(x)]$ and $\gamma^d_c(x)$, which, initially, may be taken as smooth functions of compact support. Here π^{ab} transforms as a contravariant tensor density of rank two and has dimensions M/LT, while

γ_c^d transforms as a mixed tensor and is dimensionless. With $|\eta\rangle$—called the *fiducial vector*—an as yet unspecified unit vector in $\mathfrak{H}$, we consider the set of unit vectors (in units where $\hbar = 1$) each of which is given by

$$|\pi, \gamma\rangle \equiv e^{i\int \pi^{ab}(x)\, \hat{g}_{ab}(x)\, d^3x}\, e^{-i\int \gamma_c^d(x)\, \hat{\pi}_d^c(x)\, d^3x}\, |\eta\rangle \ . \tag{4}$$

As π^{ab} and γ_c^d range over the space of smooth functions of compact support, such vectors form a set of *coherent states*.

The complex functional $\langle \pi'', \gamma'' | \pi', \gamma' \rangle$ formed by the inner product of two such coherent states will be a functional of fundamental importance in the present study of gravity. In particular, the functional $\langle \pi'', \gamma'' | \pi', \gamma' \rangle$, whatever form it takes, is manifestly a positive-definite functional that fulfills the defining condition that

$$\sum_{j,k=1}^{J} \alpha_j^*\, \alpha_k\, \langle \pi_j, \gamma_j | \pi_k, \gamma_k \rangle \geq 0 \tag{5}$$

for general sets $\{\alpha_j\}$ and $\{\pi_j, \gamma_j\}$ for any $J < \infty$. Furthermore, $\langle \pi'', \gamma'' | \pi', \gamma' \rangle$ is always a continuous functional in some natural functional topology, e.g., a topology defined by the particular expression itself [9]. As a continuous, positive-definite functional, it follows that we may adopt the expression $\langle \pi'', \gamma'' | \pi', \gamma' \rangle$ as a *reproducing kernel*, and use it to define an associated *reproducing kernel Hilbert space* $\mathcal{C}$ [10]. Let two elements of a dense set of elements in $\mathcal{C}$ be given by

$$\psi(\pi, \gamma) \equiv \sum_{j=1}^{J} \alpha_j \langle \pi, \gamma | \pi_j, \gamma_j \rangle \ , \qquad J < \infty \ ,$$
$$\phi(\pi, \gamma) \equiv \sum_{k=1}^{K} \beta_k \langle \pi, \gamma | \overline{\pi}_k, \overline{\gamma}_k \rangle \ , \qquad K < \infty \ , \tag{6}$$

where $\{\overline{\pi}_k, \overline{\gamma}_k\}_{k=1}^K$ denotes another independent set of (real) fields. These are continuous functionals of the fields π and γ. As the inner product of these two elements we adopt

$$(\psi, \phi) \equiv \sum_{j=1}^{J} \sum_{k=1}^{K} \alpha_j^*\, \beta_k \langle \pi_j, \gamma_j | \overline{\pi}_k, \overline{\gamma}_k \rangle \ . \tag{7}$$

We complete the space of functions by including the limit point of all Cauchy sequences in the norm $\|\psi\| \equiv (\psi, \psi)^{1/2}$.

The result of the above construction is the (separable) reproducing kernel Hilbert space $\mathcal{C}$ composed of bounded, continuous functionals. Moreover, the Hilbert space $\mathcal{C}$ provides an especially useful functional representation of our original Hilbert space, which was referred to as the "primary container" in P-I.

B. Choice of the fiducial vector and explicit form of the reproducing kernel

As argued in Sec. I, the representation of the basic field operators $\hat{g}_{ab}$ and $\hat{\pi}^c_d$ must be ultralocal prior to the introduction of any constraints. To fulfill this requirement, it is necessary that

$$\langle \pi'', \gamma'' | \pi', \gamma' \rangle = \exp\{-{\textstyle\int} b(x)\, d^3x\, L[\pi''(x), \gamma''(x); \pi'(x), \gamma'(x)]\} \tag{8}$$

for some dimensionless scalar function L. This function is determined by the representation of the affine field operators and the fiducial vector $|\eta\rangle$, and as minimum conditions we require that

$$\langle \eta |\, \hat{g}_{ab}(x)\, | \eta \rangle \equiv \tilde{g}_{ab}(x) \ , \tag{9}$$

$$\langle \eta |\, \hat{\pi}^c_d(x)\, | \eta \rangle \equiv 0 \ . \tag{10}$$

Here, $\tilde{g}(x) \equiv \{\tilde{g}_{ab}(x)\}$ is a fixed, smooth, positive-definite metric function determined by the choice of $|\eta\rangle$ (see [1]). Whether $\mathcal{S}$ is compact or noncompact, the choice of $\tilde{g}(x)$ will determine the topology of the space-like surfaces under consideration; if $\mathcal{S}$ is noncompact, then $\tilde{g}(x)$ also determines the asymptotic form of the space-like surfaces under consideration.

For reasons to be offered below, we choose $|\eta\rangle$ so that the overlap function of two coherent states is given (when $\hbar = 1$) by

$$\begin{aligned} &\langle \pi'', \gamma'' | \pi', \gamma' \rangle \\ &\quad = \exp\Big[-2 \int b(x)\, d^3x \\ &\quad \times \ln \Big(\frac{\det\{\frac{1}{2}[g''^{ab}(x) + g'^{ab}(x)] + \frac{1}{2} i b(x)^{-1} [\pi''^{ab}(x) - \pi'^{ab}(x)]\}}{\{\det[g''^{ab}(x)]\, \det[g'^{ab}(x)]\}^{1/2}} \Big) \Big] \ . \end{aligned} \tag{11}$$

Several comments about this basic expression are in order.

Initially, regarding (11), we observe that γ'' and γ' do *not* appear in the explicit functional form given. In particular, the smooth matrix γ has been replaced by the smooth matrix g which is defined at every point by

$$g(x) \equiv e^{\gamma(x)/2}\,\tilde{g}(x)\,e^{\gamma(x)^T/2} \equiv \{g_{ab}(x)\}\ , \tag{12}$$

where $\gamma(x)^T$ denotes the transpose of the matrix $\gamma(x)$. Observe that the so-defined matrix $\{g_{ab}(x)\}$ is manifestly positive definite for all x. The map $\gamma \to g$ is clearly many-to-one since γ has *nine* independent variables at each point while g, which is symmetric, has only *six*. In view of this functional dependence *we may denote the given functional in (11) by* $\langle\pi'', g''|\pi', g'\rangle$, *and henceforth we shall adopt this notation exclusively.*

Single affine matrix degree of freedom

An elementary example of the notational change from γ to g may be seen quite directly in what occurs for a *single* affine matrix degree of freedom (in contrast to a field of such degrees of freedom). To that end, and following Ref. 11 closely, we introduce the Lie algebra for affine matrix self-adjoint operator degrees of freedom composed of the symmetric 3×3 matrix $\{\sigma_{ab}\}$ and the 3×3 matrix $\{\kappa^c_d\}$, which together obey the affine commutation relations [cf., (1)]

$$\begin{aligned}
[\kappa^a_b, \kappa^c_d] &= i\,\tfrac{1}{2}\,(\delta^c_b\kappa^a_d - \delta^a_d\kappa^c_b)\ ,\\
[\sigma_{ab}, \kappa^c_d] &= i\,\tfrac{1}{2}\,(\delta^c_a\sigma_{db} + \delta^c_b\sigma_{ad})\ ,\\
[\sigma_{ab}, \sigma_{cd}] &= 0\ .
\end{aligned} \tag{13}$$

We choose the faithful, irreducible representation for which the operator matrix $\{\sigma_{ab}\}$ is symmetric and positive definite, and which is unique up to unitary equivalence. Furthermore, we choose a representation which diagonalizes $\{\sigma_{ab}\}$ as $k \equiv \{k_{ab}\}$, which we refer to as the k-representation. In the associated L^2 representation space, and for arbitrary real matrices $F = \{F^{ab}\}$, $F^{ba} = F^{ab}$, and $B = \{B^c_d\}$, it follows that

$$\begin{aligned}
U[F,B]\,\psi(k) &\equiv e^{iF^{ab}\sigma_{ab}}\,e^{-iB^c_d\kappa^d_c}\,\psi(k)\\
&= (\det[S])^2\,e^{iF^{ab}k_{ab}}\,\psi(SkS^T)\ ,
\end{aligned} \tag{14}$$

where $S \equiv e^{-B/2} = \{S^a_b\}$ and $(SkS^T)_{ab} \equiv S^c_a k_{cd} S^d_b$. The given transformation is unitary within the inner product defined by

$$\int_+ \psi(k)^* \psi(k)\, dk \;, \tag{15}$$

where $dk \equiv \Pi_{a\le b}\, dk_{ab}$, and the "+" sign denotes an integration over only that part of the six-dimensional k-space where the elements form a symmetric, positive-definite matrix, $\{k_{ab}\} > 0$. To define coherent states we choose an extremal weight vector,

$$\eta(k) \equiv C\,(\det[k])^{\beta-1}\, e^{-\beta\,\mathrm{tr}[\tilde{G}^{-1}k]} \;, \tag{16}$$

where $\beta > 0$, $\tilde{G} = \{\tilde{G}_{ab}\}$ is a fixed positive-definite matrix, C is determined by normalization, and tr denotes the trace. This choice leads to the expectation values

$$\langle\eta|\,\sigma_{ab}\,|\eta\rangle = \int_+ \eta(k)^*\, k_{ab}\eta(k)\, dk = \tilde{G}_{ab} \;, \tag{17}$$

$$\langle\eta|\,\kappa^c_d\,|\eta\rangle = \int_+ \eta(k)^*\, \kappa^c_d \eta(k)\, dk = 0 \;. \tag{18}$$

In the k-representation, it follows that the affine matrix coherent states are given by

$$\langle k|F,B\rangle \equiv C\,(\det[S])^2\,(\det[SkS^T])^{\beta-1}\, e^{i\,\mathrm{tr}[Fk]}\, e^{-\beta\,\mathrm{tr}[\tilde{G}^{-1}SkS^T]} \;. \tag{19}$$

Observe that what really enters the functional argument is the positive-definite matrix $G^{-1} \equiv S^T\tilde{G}^{-1}S$ where we set $G \equiv \{G_{ab}\}$. Thus without loss of generality we can drop the label B (or equivalently S) and replace it with G. Hence the affine matrix coherent states become

$$\langle k|F,G\rangle \equiv C'\,(\det[G^{-1}])^{\beta}\,(\det[k])^{\beta-1}\, e^{i\,\mathrm{tr}[Fk]}\, e^{-\beta\,\mathrm{tr}[G^{-1}k]} \;, \tag{20}$$

where $C' = C\,(\det[\tilde{G}])^{\beta}$ is a new normalization constant. It is now straightforward to determine that

$$\begin{aligned} \langle F'',G''|F',G'\rangle &= \int_+ \langle F'',G''|k\rangle\,\langle k|F',G'\rangle\, dk \\ &= \Big[\frac{\{\det[G''^{-1}]\,\det[G'^{-1}]\}^{1/2}}{\det\{\frac{1}{2}[(G''^{-1}+G'^{-1}) + i\beta^{-1}(F''-F')]\}}\Big]^{2\beta} \;. \end{aligned} \tag{21}$$

In arriving at this result, we have used normalization of the coherent states to eliminate the constant C'.

Lattice construction

Suppose now that we consider an independent lattice of such matrix degrees of freedom and build the corresponding coherent state overlap as the product of ones just like (21). Let $\mathbf{n}$ label a lattice site and let $\mathbf{n} \in \mathbf{N}$, which in turn is a finite subset of $\mathbb{Z}^3$. In that case the coherent state overlap is given by

$$\begin{aligned}&\langle F'', G''|F', G'\rangle_{\mathbf{N}} \\ &\quad = \prod_{\mathbf{n}\in\mathbf{N}} \Big[\frac{\{\det[G''^{-1}_{[\mathbf{n}]}]\det[G'^{-1}_{[\mathbf{n}]}]\}^{1/2}}{\det\{\frac{1}{2}[(G''^{-1}_{[\mathbf{n}]} + G'^{-1}_{[\mathbf{n}]}) + i\beta^{-1}_{[\mathbf{n}]}(F''_{[\mathbf{n}]} - F'_{[\mathbf{n}]})]\}}\Big]^{2\beta_{[\mathbf{n}]}} . \quad (22)\end{aligned}$$

As our next step we wish to take a limit in which the number of independent matrix degrees of freedom tends to infinity in such a way that not only does the lattice size diverge but also the lattice spacing tends to zero so that, loosely speaking, the lattice points approach the points of the space $\mathcal{S}$. In order for the limit to be nonzero, it is necessary that the exponent $\beta_{[\mathbf{n}]} \to 0$ in a suitable way. To that end we set

$$\beta_{[\mathbf{n}]} \equiv b_{[\mathbf{n}]}\,\Delta\ , \quad (23)$$

where Δ has the dimensions L^3, and thus $b_{[\mathbf{n}]}$ has the dimensions L^{-3}. In addition we need to let $F^{ab}_{[\mathbf{n}]} \equiv \pi^{ab}_{[\mathbf{n}]}\Delta$, and we rename $G_{ab\,[\mathbf{n}]}$ as $g_{ab\,[\mathbf{n}]}$ and call the matrix elements of $G^{-1}_{[\mathbf{n}]}$ by $g^{ab}_{[\mathbf{n}]}$. With these changes (22) becomes

$$\begin{aligned}&\langle \pi'', g''|\pi', g'\rangle_{\mathbf{N}} \\ &\quad \equiv \prod_{\mathbf{n}\in\mathbf{N}} \Big[\frac{\{\det[g''^{ab}_{[\mathbf{n}]}]\det[g'^{ab}_{[\mathbf{n}]}]\}^{1/2}}{\det\{\frac{1}{2}[(g''^{ab}_{[\mathbf{n}]} + g'^{ab}_{[\mathbf{n}]}) + i\,b^{-1}_{[\mathbf{n}]}(\pi''^{ab}_{[\mathbf{n}]} - \pi'^{ab}_{[\mathbf{n}]})]\}}\Big]^{2\,b_{[\mathbf{n}]}\,\Delta} . \quad (24)\end{aligned}$$

Finally, we take the limit as described above and the result is given by (11), namely,

$$\begin{aligned}&\langle \pi'', g''|\pi', g'\rangle \\ &\quad = \exp\Big[-2\int b(x)\,d^3x \\ &\qquad \times \ln\Big(\frac{\det\{\frac{1}{2}[g''^{ab}(x) + g'^{ab}(x)] + \frac{1}{2}ib(x)^{-1}[\pi''^{ab}(x) - \pi'^{ab}(x)]\}}{\{\det[g''^{ab}(x)]\ \det[g'^{ab}(x)]\}^{1/2}}\Big)\Big] . \quad (25)\end{aligned}$$

In this way we see how the continuum result may be obtained as a limit starting from a collection of independent affine matrix degrees of freedom. The necessity of ending with an integral over the space $\mathcal{S}$ has directly led to the requirement that we introduce the scalar density $b(x)$.

C. Additional arguments favoring $b(x)$

As a further general comment about (11) or (25) we observe that $\langle \pi'', g'' | \pi', g' \rangle$ is *invariant* under general (smooth, invertible) coordinate transformations $x \rightarrow \overline{x} = \overline{x}(x)$, and we say that the given expression characterizes a *diffeomorphism invariant realization* of the affine field operators. This property holds, in part, because $b(x)$ transforms as a scalar density in both places that it appears. Thus $b(x)$, which has the dimensions of L^{-3}, plays an essential dimensional and transformational role in each place that it appears. Note that if $\hbar$ is explicitly introduced into (25) or (11), it appears only in the change $[\pi''^{ab}(x) - \pi'^{ab}(x)] \rightarrow [\pi''^{ab}(x) - \pi'^{ab}(x)]/\hbar$. If one insisted on building an acceptable ultralocal positive-definite functional using only $\pi''^{ab}(x)$, $g''^{ab}(x)$, $\pi'^{ab}(x)$, $g'^{ab}(x)$, and $\hbar$, then that construction would not appear to be possible simply on dimensional grounds.

As another argument for the appearance of $b(x)$, we observe that (25) involves not only the c-number fields π and g (or γ), but the particular representation of the local operators $\hat{g}_{ab}$ and $\hat{\pi}^c_d$ as well as the choice of the fiducial vector $|\eta\rangle$. It is entirely natural that the function $b(x)$ may emerge as a needed functional parameter in defining the operator representation and/or the vector $|\eta\rangle$, and this property is explicitly illustrated in P-I.

As a final argument for the appearance of the scalar density $b(x)$, we briefly recall properties of the local operator product for ultralocal affine field operators [1]. In particular, the formal local product reads

$$\hat{g}_{ab}(x)\,\hat{g}_{cd}(x) = \delta(x,x)\,\hat{E}_{abcd}(x) + l.s.t.\ . \tag{26}$$

Here $\hat{E}_{abcd}(x)$ is a local fourth-order covariant tensor density operator of weight -1, $\delta(x,x)$ is a divergent multiplier with dimensions L^{-3}, which arises when the "scalar density" delta function $\delta(x,y)$ is evaluated at coincident points, and $l.s.t.$ denotes "less singular terms". Before adopting the proper local operator product, we introduce a scalar density $b(x)$, $0 < b(x) < \infty$, with dimensions L^{-3}, and consider

$$\hat{g}_{ab}(x)\,\hat{g}_{cd}(x) = b(x)\,[b(x)^{-1}\,\delta(x,x)]\,\hat{E}_{abcd}(x) + l.s.t.\ . \tag{27}$$

Finally, we choose

$$[\hat{g}_{ab}(x)\,\hat{g}_{cd}(x)]_R \equiv b(x)\,\hat{E}_{abcd}(x) \;\; (= \hat{g}_{ab}(x)\,\hat{g}_{cd}(x)/[b(x)^{-1}\delta(x,x)]) \tag{28}$$

as the proper renormalized (subscript R) local operator product. (Limits involving test function sequences offer a mathematically precise construction.) The given choice leads to a local operator that transforms as a tensor in the natural fashion and, moreover, carries the natural engineering dimensions. To achieve this desirable property in a local product *requires* the introduction of an auxiliary scalar density $b(x)$.

For additional properties regarding local products of the relevant affine field operators, see P-I. Essentially, all these properties are almost entirely based on the analysis of local operator products in scalar ultralocal field theories, an analysis which is described in detail, for example, in [12].

We have offered several reasons for the appearance of the scalar density $b(x)$ at the present stage of the analysis. However, we emphasize once again that $b(x)$ will disappear when the constraints are fully enforced, whatever choice was originally made. An example of the process by which this fundamental transformation takes place is presented in Sec. IV.

D. Functional integral representation for the coherent state overlap functional

For further analysis, especially when we take up the issue of introducing the constraints in Sec. III, it is useful to introduce an alternative representation of the functional expression (25). The alternative representation we have in mind is that of a specific functional integral, which, indeed, has already been introduced in P-I. That such a representation should exist is an immediate consequence of the fact that (25) fulfills a complex polarization condition, which then leads to (25) being annihilated by a negative, second-order functional derivative operator. Exponentiating this operator, times a parameter $\nu > 0$, letting the resultant operator act on general functionals of π'' and g'', and including any necessary ν-dependent prefactor, will, in the limit $\nu \to \infty$, lead to a dense set of functions in the reproducing kernel Hilbert space $\mathcal{C}$. Alternatively, letting the same operator act on a suitable δ-functional will lead to the expression (25). The functional integral of interest arises in this last expression by introducing the analog of a Feynman-Kac-Stratonovich

representation. The mathematics behind these foregoing several sentences is well illustrated in P-I for both a simple, single affine degree-of-freedom example as well as for the affine field theory.

The result of the operations outlined above leads to a functional integral representation for $\langle\pi'', g''|\pi', g'\rangle$ in (25), which is given (for $\hbar = 1$) by

$$\begin{aligned}
&\langle\pi'', g''|\pi', g'\rangle \\
&\quad= \exp\Big[-2\int b(x)\, d^3\!x \\
&\qquad\times \ln\Big(\frac{\det\{\frac{1}{2}[g''^{ab}(x)+g'^{ab}(x)]+\frac{1}{2}ib(x)^{-1}[\pi''^{ab}(x)-\pi'^{ab}(x)]\}}{\{\det[g''^{ab}(x)]\,\det[g'^{ab}(x)]\}^{1/2}}\Big)\Big] \\
&\quad= \lim_{\nu\to\infty}\overline{\mathcal{N}}_\nu \int \exp[-i\textstyle\int g_{ab}\dot{\pi}^{ab}\, d^3\!x\, dt] \\
&\qquad\times \exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da}+b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\, d^3\!x\, dt\} \\
&\qquad\quad\times\Pi_{x,t}\,\Pi_{a\le b}\, d\pi^{ab}(x,t)\, dg_{ab}(x,t)\ . \qquad (29)
\end{aligned}$$

Here, because of the way the new independent variable t appears on the right-hand side of this expression, it is natural to interpret t, $0 \le t \le T$, $T > 0$ as coordinate "time". The fields on the right-hand side all depend on space and time, i.e., $g_{ab} = g_{ab}(x,t)$, $\dot{g}_{ab} = \partial g_{ab}(x,t)/\partial t$, etc., and, importantly, the integration domain of the formal measure is strictly limited to the domain where $\{g_{ab}(x,t)\}$ is a positive-definite matrix for all x and t. For the boundary conditions, we have $\pi'^{ab}(x) \equiv \pi^{ab}(x,0)$, $g'_{ab}(x) \equiv g_{ab}(x,0)$, as well as $\pi''^{ab}(x) \equiv \pi^{ab}(x,T)$, $g''_{ab}(x) \equiv g_{ab}(x,T)$ for all x. Observe that the right-hand side holds for *any* T, $0 < T < \infty$, while the middle term is *independent of* T *altogether*.

Although the functional integral on the right-hand side is formal it nevertheless conveys a great deal of information. Let us first examine it from a dimensional and transformational standpoint. As presented, $\hbar = 1$; to see where $\hbar$ would appear we may simply replace each π^{ab} by $\pi^{ab}/\hbar$. With ν having the dimensions of T^{-1} and $\overline{\mathcal{N}}_\nu$ absorbing any remaining dimensions from the formal measure, then the right-hand side of (29) is dimensionally satisfactory. From the point of view of (formal) transformations under coordinate changes, it is clear, with $\overline{\mathcal{N}}_\nu$ transforming appropriately, that the right-hand side is formally invariant under coordinate transformations involving the spatial coordinates alone. **(Remark:** A discussion about transformations of the right-hand side under spatially dependent transformations of the time coordinate has been given in P-I and is not repeated here. It is clear that the

result of the limit $\nu \to \infty$ on the right-hand side must be invariant under all such transformations simply because the middle term is independent of the time variable altogether!**)**

As presented—and indeed as originally derived—the *result* of the functional integral (the middle term) came before the functional integral *representation* of that result (the right-hand side). However, we can also interpret (29) in the opposite order, that is, to presume that the functional integral (right-hand side) is primary and that the answer (the middle term) is the result of evaluating the functional integral. This perspective encourages us to examine the expression in the integrand of the functional integral somewhat more carefully from a traditional standpoint. We first observe that the formal, flat part of the measure has the expected appearance of the *canonical measure* for a conventional, canonical functional integral quantization of gravity. The phase factor contains an acceptable classical symplectic potential term in a formal functional integral for gravity in which the rest of the classical action—the terms involving the constraints—are absent. **(Remark:** This characterization is, of course, quite appropriate since we still are in the first phase of our dual approach: quantize first, reduce second. The patient reader will be rewarded with the addition of the expected constraints and Lagrange multiplier terms in the next section.**)** The second, ν-dependent factor in the integrand serves as a *regularizing factor* for the functional integral. Formally, as $\nu \to \infty$, such a factor disappears from the integrand leaving the expected (pre-constraint) formal functional integral integrand, such as it is. However, the ν-dependent term plays a fundamentally important role within the integral itself since it *literally serves to define the functional integral.*

It is important to make this last point quite clear, and for that purpose we temporarily discuss the formal expression (with $\hbar = 1$ again)

$$\begin{aligned} &d\mu^\nu(\pi, g) \\ &\quad = \mathcal{M}_\nu \, \exp\{-(1/2\nu){\textstyle\int}[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da} + b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\, d^3\!x\, dt\} \\ &\qquad\quad \times \Pi_{x,t}\, \Pi_{a\le b}\, d\pi^{ab}(x,t)\, dg_{ab}(x,t)\ . \end{aligned} \tag{30}$$

We assert that for fixed $b(x)$, $0 < b(x) < \infty$ and fixed ν, $0 < \nu < \infty$, this expression characterizes a *bona fide, countably-additive, positive measure*, μ^ν, on the space of generalized functions $\pi^{ab} = \pi^{ab}(x,t)$ and $g_{ab} = g_{ab}(x,t)$, where, for any nonvanishing u^a and any nonvanishing, nonnegative test function $f(x) \ge 0$, the positive-definite matrix condition $\int f(x)\, u^a\, g_{ab}(x,t)\, u^b\, d^3\!x > 0$

holds for (almost) all t, $0 < t < T$. The fields π^{ab} and g_{ab} satisfy the boundary conditions at $t = 0$ and $t = T$ given previously. The factor $\mathcal{M}_\nu$ is adjusted so that the measures μ^ν form a semi-group with respect to combining time intervals, e.g., $0 \to T$, $T > 0$, and then $T \to T + T'$, $T' > 0$, being equivalent to $0 \to T + T'$. If $\{h_p(x)\}_{p=1}^\infty$ denotes an orthonormal set of test functions defined so that

$$\textstyle\int h_p(x)\, h_q(x)\, b(x)\, d^3\!x = \delta_{pq}\ , \tag{31}$$

$$b(x)\, \Sigma_{p=1}^\infty h_p(x)\, h_p(y) = \delta(x, y)\ , \tag{32}$$

then we assert that there exist finite, nonzero constants N_P^ν for all ν and all $P \in \{1, 2, 3, \ldots\}$ such that

$$N_P^\nu \int \exp[-i\Sigma_{p=1}^P \textstyle\int g_{ab(p)}(t)\, \dot{\pi}_{(p)}^{ab}(t)\, dt]\, d\mu^\nu(\pi, g) \tag{33}$$

is well defined. In this expression,

$$g_{ab(p)}(t) \equiv \textstyle\int h_p(x)\, g_{ab}(x, t)\, b(x)\, d^3\!x\ , \tag{34}$$

$$\dot{\pi}_{(p)}^{ab}(t) \equiv \textstyle\int h_p(x)\, \dot{\pi}^{ab}(x, t)\, d^3\!x\ . \tag{35}$$

Moreover, the set of constants $\{N_P^\nu\}$ may be chosen so that

$$\begin{aligned} &\langle \pi'', g'' | \pi', g' \rangle \\ &\quad \equiv \lim_{P\to\infty} \lim_{\nu\to\infty} N_P^\nu \int \exp[-i\Sigma_{p=1}^P \textstyle\int g_{ab(p)}(t)\, \dot{\pi}_{(p)}^{ab}(t)\, dt]\, d\mu^\nu(\pi, g)\ . \end{aligned} \tag{36}$$

This is one of the ways that the formal functional integral (29) can be given a rigorous meaning.

There is another way to give rigorous meaning to (29) that we would also like to discuss. In this procedure we use a spatial lattice but keep the time variable t continuous. This regularization scheme takes us back to the lattice construction given earlier [cf., (24)], except now we add a phase-space path integral representation as well. For present purposes we again introduce the symbol Δ, with dimensions L^3, to denote a uniformly (coordinate) sized, small spatial cell. Then the lattice regularized path integral expression for $\langle \pi'', g'' | \pi', g' \rangle_{\mathbf{N}}$ in (24) is given by [13]

$$\begin{aligned} &\overline{\mathcal{N}}_\nu^{\mathbf{N}} \int e^{-i\Sigma_{\mathbf{n}} \int g_{ab[\mathbf{n}]}\, \dot{\pi}_{[\mathbf{n}]}^{ab}\, \Delta\, dt} \\ &\quad \times \exp\{-\tfrac{1}{2\nu} \Sigma_{\mathbf{n}} \textstyle\int [b_{[\mathbf{n}]}^{-1} g_{ab[\mathbf{n}]} g_{cd[\mathbf{n}]} \dot{\pi}_{[\mathbf{n}]}^{bc} \dot{\pi}_{[\mathbf{n}]}^{da} + b_{[\mathbf{n}]} g_{[\mathbf{n}]}^{ab} g_{[\mathbf{n}]}^{cd} \dot{g}_{bc[\mathbf{n}]} \dot{g}_{da[\mathbf{n}]}]\, \Delta\, dt\} \\ &\qquad \times \Pi_{\mathbf{n},t}\, \Pi_{a\le b}\, d\pi_{[\mathbf{n}]}^{ab}(t)\, dg_{ab[\mathbf{n}]}(t)\ , \end{aligned} \tag{37}$$

where $\mathbf{n} \in \mathbf{N}$, which itself is a finite subset of $\mathbb{Z}^3$. Here $\pi^{ab}_{[\mathbf{n}]}$, $g_{ab[\mathbf{n}]}$, and $b_{[\mathbf{n}]}$ represent average field values in the cell $\mathbf{n}$, and $b^{-1}_{[\mathbf{n}]} = 1/b_{[\mathbf{n}]}$. Next, we observe that there is a countably-additive, pinned, Brownian-motion measure formally defined by

$$\begin{aligned} & d\mu^{\nu}_{\mathbf{N}}(\pi, g) \\ \equiv \mathcal{N}^{\mathbf{N}}_{\nu} \, & \exp\{-\tfrac{1}{2\nu}\Sigma_{\mathbf{n}}\textstyle\int[b^{-1}_{[\mathbf{n}]} g_{ab[\mathbf{n}]} g_{cd[\mathbf{n}]} \dot{\pi}^{bc}_{[\mathbf{n}]} \dot{\pi}^{da}_{[\mathbf{n}]} + b_{[\mathbf{n}]} g^{ab}_{[\mathbf{n}]} g^{cd}_{[\mathbf{n}]} \dot{g}_{bc[\mathbf{n}]} \dot{g}_{da[\mathbf{n}]}]\,\Delta\,dt\} \\ & \qquad \times \Pi_{\mathbf{n},t}\,\Pi_{a\le b}\, d\pi^{ab}_{[\mathbf{n}]}(t)\, dg_{ab[\mathbf{n}]}(t) \end{aligned} \tag{38}$$

so that (37) becomes

$$\overline{N}^{\mathbf{N}}_{\nu} \int e^{-i\Sigma_{\mathbf{n}}\int g_{ab[\mathbf{n}]}\,\dot{\pi}^{ab}_{[\mathbf{n}]}\,\Delta\,dt}\, d\mu^{\nu}_{\mathbf{N}}(\pi, g) \;, \tag{39}$$

where $\{\overline{N}^{\mathbf{N}}_{\nu}\}$ is a set of finite constants. Moreover, these constants may be chosen so that

$$\langle \pi'', g'' | \pi', g' \rangle = \lim_{\mathbf{N}\to\infty} \lim_{\nu\to\infty} \overline{N}^{\mathbf{N}}_{\nu} \int e^{-i\Sigma_{\mathbf{n}}\int g_{ab[\mathbf{n}]}\,\dot{\pi}^{ab}_{[\mathbf{n}]}\,\Delta\,dt}\, d\mu^{\nu}_{\mathbf{N}}(\pi, g) \;. \tag{40}$$

Here, in the last step, the limit $\mathbf{N} \to \infty$ means that $\Delta \to 0$ and $\mathbf{N} \to \mathbb{Z}^3$ in such a way that all points $x \in \mathcal{S}$ are reached in a natural way. Observe that the presence of the continuous-time regularization factor in the formal functional integral for the entire space has controlled the spatial lattice regularization in a clear and natural fashion; although possible to introduce, no temporal lattice has been required to obtain a well-defined expression.

The present usage of well-defined, phase-space measures to define functionals integrals in the limit that the "diffusion constant" parameter (ν) diverges is part of the general program of *continuous-time regularization* [14]. It is noteworthy that the use of a suitable *phase-space metric* to control the Brownian motion paths invariably leads to a coherent-state representation for the resultant quantum amplitude. These remarks conclude our brief excursion into a rigorous discussion of the functional integrals of present interest.

Another prospective regularization

Let us return to the formal functional integral (29) and examine that expression with regard to the scalar density $b(x)$. Superficially, in the limit $\nu \to \infty$

in which the regularizing term proportional to $(1/2\nu)$ in the exponent of the integrand formally vanishes, one might naively expect that the result of the integral would be independent of the function $b(x)$. But, no, that naive expectation is false since the result of that integral and subsequent limit, i.e., the middle term of (29), evidently depends importantly on $b(x)$. At first glance, that dependence seems highly unnatural. It may be thought—as the author did for a number of years [15]—that an alternative regularization expression may be more "natural" and would therefore be preferable. In particular, it was thought that the expression

$$\exp\{-(1/2\nu)\textstyle\int[g^{-1/2}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da}+g^{1/2}g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\,d^3x\,dt\}\;, \tag{41}$$

where $g=\det[g_{ab}]$, and which involves a different phase-space metric, was "better". However, in light of the discussion in the present paper, it is now evident that this expression is not even dimensionally consistent! This defect could be rectified by the introduction of a positive constant, which we may call $\tilde{b}$, that stands in the place of the present b (next to $g^{1/2}$) and carries the dimensions L^{-3}. Indeed, we could also introduce in place of $\tilde{b}$ a positive scalar function $\tilde{b}(x)$ with dimensions L^{-3}. In any case, the failure of (41) purely on dimensional grounds, and the necessity thereby of introducing some sort of auxiliary dimensioned parameter (or function) surely renders (41) far less "natural" than had been previously assumed.

As another possible argument against (41), we note that if that form of a proposed regularization was used in an expression like (29) in place of the present form of regularization, there is absolutely no guarantee that the result will describe a reproducing kernel with other than a *one-dimensional reproducing kernel Hilbert space*, which is the general result for a "random" choice of phase-space metric. The fact that the present form of (29) generates a suitable, infinite-dimensional reproducing kernel Hilbert space is a fundamentally important feature, which, in our case, is a consequence of having started with an appropriate reproducing kernel to begin with.

Although the author has not foreclosed any possible interest in a $\tilde{b}$ modified version of the regularization (41), all present indications point to the version (30) that features the scalar density $b(x)$. This shift of allegiance has also been bolstered by the realization that the role of $b(x)$ is confined to the initial phase of quantization and that $b(x)$ will disappear entirely after the constraints are fully enforced. On the strength of this argument, it is the

version based on (30) and not (41) that is analyzed in the remainder of the present paper.

III. INTRODUCTION OF THE GRAVITATIONAL CONSTRAINTS

A. Key principles in heuristic form

There are four gravitational constraint functions, the three diffeomorphism constraints

$$H_a(x) = -2\pi^b_{a|b}(x) \ , \tag{42}$$

where "$_|$" denotes covariant differentiation with respect to the spatial metric g_{ab}, and the Hamiltonian constraint, which, in suitable units (i.e., $c^3/G = 16\pi$), reads

$$H(x) = g(x)^{-1/2}[\,\pi^a_b(x)\pi^b_a(x) - \tfrac{1}{2}\pi^a_a(x)\pi^b_b(x)\,] + g(x)^{1/2\,(3)}R(x) \ , \tag{43}$$

where $g(x) \equiv \det[g_{ab}(x)]$ and ${}^{(3)}R(x)$ denotes the scalar curvature derived from the spatial metric [16]. Classically, these constraint functions vanish, and the region in phase space on which they vanish is called the *constraint hypersurface.*

It is instructive to evaluate the classical Poisson brackets between the constraint fields. For this purpose, we enlist only the basic nonvanishing Poisson bracket given by

$$\{g_{ab}(x), \pi^{cd}(y)\} = \tfrac{1}{2}(\delta^c_a\delta^d_b + \delta^c_b\delta^d_a)\,\delta(x,y) \ . \tag{44}$$

It follows that

$$\{H_a(x), H_b(y)\} = \delta_{,a}(x,y)\,H_b(x) - \delta_{,b}(x,y)\,H_a(x) \ , \tag{45}$$

$$\{H_a(x), H(y)\} = \delta_{,a}(x,y)\,H(x) \ , \tag{46}$$

$$\{H(x), H(y)\} = \delta_{,a}(x,y)\,g^{ab}(x)\,H_b(x) \ . \tag{47}$$

In these expressions, $\delta_{,a}(x,y) \equiv \partial\delta(x,y)/\partial x^a$, which transforms as a "vector density". It is clear that the Poisson brackets of the constraints vanish

on the constraint hypersurface because the right-hand sides of (45)–(47) all vanish there, i.e., when $H_a(x) = 0 = H(x)$ for all $x \in \mathcal{S}$. This vanishing property of the Poisson brackets is characteristic of *first-class constraints*. Often, the Poisson bracket structure of constraints is that of a Lie bracket in which case such constraints are referred to as closed first-class constraints. In order for the Poisson bracket structure to be that of a Lie bracket, it is necessary that the coefficients of the constraints on the right-hand side involve no dynamical variables. Taken by themselves, we note from (45) that the three diffeomorphism constraint functions form a set of closed first-class constraints. However, because of the last equation, (47), it is clear that the complete set of four gravitational constraint functions do *not* have a Poisson structure which is that of a Lie algebra, and consequently the gravitational constraints are said to form an open first-class system of constraints. Such a situation does not automatically imply trouble in the corresponding quantum theory, but significant difficulties do arise in a number of cases. Quantum gravity is one of those cases.

Let us proceed formally in order to see the essence of the problem. Suppose that $\mathcal{H}_a(x)$ and $\mathcal{H}(x)$ represent local self-adjoint constraint operators for the gravitational field. Standard calculations lead to the commutation relations (with $\hbar = 1$)

$$[\mathcal{H}_a(x), \mathcal{H}_b(y)] = i\,[\delta_{,a}(x,y)\,\mathcal{H}_b(x) - \delta_{,b}(x,y)\,\mathcal{H}_a(x)]\ , \tag{48}$$

$$[\mathcal{H}_a(x), \mathcal{H}(y)] = i\,\delta_{,a}(x,y)\,\mathcal{H}(x)\ , \tag{49}$$

$$[\mathcal{H}(x), \mathcal{H}(y)] = i\,\tfrac{1}{2}\,\delta_{,a}(x,y)[\,\hat{g}^{ab}(x)\,\mathcal{H}_b(x) + \mathcal{H}_b(x)\,\hat{g}^{ab}(x)\,]\ , \tag{50}$$

where to ensure the Hermitian character we have symmetrized the right-hand side of the last expression. In the usual Dirac approach to constraints alluded to in Sec. I, one asks that $\Phi_\alpha|\psi\rangle_{phys} = 0$ for all constraints. If we assert that $\mathcal{H}_a(x)|\psi\rangle_{phys} = 0$ and $\mathcal{H}(x)|\psi\rangle_{phys} = 0$, then consistency holds for the first two sets of constraint commutators, but not for the third commutator in virtue of the fact that it is almost surely the case that $\hat{g}^{ab}(x)\,|\psi\rangle_{phys} \notin \mathfrak{H}_{phys}$, even if it were smeared. The expected behavior is somewhat like that of the single degree-of-freedom example where $Q|\psi\rangle = 0$ and $P|\psi\rangle = 0$ imply for standard Heisenberg operators that $[Q,P]|\psi\rangle = i|\psi\rangle = 0$, i.e., there are no nonvanishing solutions. This behavior is characteristic of second-class constraints, and as a consequence of our discussion we are led to conclude, from a quantum mechanical standpoint, that part of the gravitational con-

straints are *second-class constraints* [17]. For the projection operator method of constrained system quantization, however, second-class constraints cause no special difficulty and, in particular, they are treated in just the same way as first-class constraints, as already noted in Sec. I. **(Remark:** Some researchers prefer to modify the theory so as to eliminate the second-class nature of the gravitational constraints. Instead, we accept the second-class constraints for what they are.**)**

Assuming that the constraint fields $\mathcal{H}_a(x)$ and $\mathcal{H}(x)$ are local self-adjoint operators, we could—as one of several different alternatives—proceed as follows. Initially, besides the real, orthonormal set of test functions $\{h_p(x)\}$ introduced in Sec. II, let us introduce an additional set of real test functions $\{f^a_{p\,A}(x)\}$, $A \in \{1,2,3\}$, with the following properties:

$$b(x)\Sigma^{\infty}_{p=1}\Sigma^3_{A=1}f^a_{p\,A}(x)\,f^b_{p\,A}(y) = \tilde{g}^{ab}(x)\,\delta(x,y)\;, \tag{51}$$

$$\textstyle\int f^a_{p\,A}(x)\,f_{q\,B\,a}(x)\,b(x)\,d^3\!x = \delta_{pq}\,\delta_{AB}\;, \tag{52}$$

where $f_{q\,B\,a}(x) \equiv \tilde{g}_{ab}(x)\,f^b_{q\,B}(x)$. Regarding the set of functions $\{f^a_{p\,A}(x)\}$, the index A is a dreibein index, while the index a is a three-space vector index. With the help of these sets of expansion functions, let us introduce

$$\mathcal{H}_{(p)\,A} \equiv \textstyle\int f^a_{p\,A}(x)\,\mathcal{H}_a(x)\,d^3\!x\;, \tag{53}$$

$$\mathcal{H}_{(p)} \equiv \textstyle\int h_p(x)\,\mathcal{H}(x)\,d^3\!x\;, \tag{54}$$

each for $1 \le p < \infty$, and similarly for other vector and scalar functions. In this form as well, part of the constraint operators are second class. To accomodate all these constraints we introduce a set of projection operators defined for all $P \in \{1,2,3,\ldots\}$ and given by

$$\mathbb{E}_P \equiv \mathbb{E}(X^2_P \le \delta(\hbar)^2)\;, \tag{55}$$

$$X^2_P \equiv \Sigma^P_{p=1}\,2^{-p}\,[\,\Sigma^3_{A=1}\,(\mathcal{H}_{(p)\,A})^2 + \mathcal{H}^2_{(p)}\,]\;. \tag{56}$$

As defined, the projection operators $\mathbb{E}_P$ are regularized and they serve to define regularized physical Hilbert spaces $\mathfrak{H}_{phys} \equiv \mathbb{E}_P\,\mathfrak{H}$. These regularized physical Hilbert spaces may, in turn, be characterized by their own reproducing kernels

$$\langle\pi'',g''|\,\mathbb{E}_P\,|\pi',g'\rangle\;, \tag{57}$$

and the regularized physical Hilbert spaces themselves may, therefore, be represented by the associated reproducing kernel Hilbert spaces.

The final step in the present construction procedure would involve suitable limits to remove the regularizations. More familiar procedures to enforce the constraints are discussed in Sec. IV.

B. Functional integral representation for the relevant projection operators

In an earlier work [5], we have presented a very general procedure to construct the projection operator $\mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2)$ by means of a universal functional integral procedure. In particular, it follows that

$$\mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2) = \int \mathbf{T}\, e^{-i\int \lambda^\alpha(t)\,\Phi_\alpha\, dt}\, \mathcal{D}R(\lambda)\ , \tag{58}$$

where $\mathbf{T}$ denotes the time-ordering operator, $\{\lambda^\alpha(t)\}_{\alpha=1}^A$, $0 \le t < T$, denotes a set of c-number "Lagrange multiplier" functions, and $\mathcal{D}R(\lambda)$ denotes a formal measure on such functions. A suitable measure R may be determined as follows: First, introduce a Gaussian integral over the set $\{\lambda^\alpha(t)\}$ so that

$$\mathcal{N}_\gamma \int \mathbf{T}\, e^{-i\int \lambda^\alpha(t)\,\Phi_\alpha\, dt}\, e^{(i/4\gamma)\int \Sigma_\alpha \lambda^\alpha(t)^2\, dt}\, \mathcal{D}\lambda = e^{-i\gamma\,\Sigma_\alpha \Phi_\alpha^2}\ . \tag{59}$$

Second, and last, integrate over γ according to the rule

$$\lim_{\zeta\to 0^+} \lim_{L\to\infty} \int_{-L}^{L} \frac{\sin\{\gamma[\delta(\hbar)^2+\zeta]\}}{\pi\gamma}\, e^{-i\gamma\,\Sigma_\alpha \Phi_\alpha^2}\, d\gamma = \mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2)\ . \tag{60}$$

The inclusion of the variable ζ and the limit $\zeta \to 0^+$ ensures that we include the equality sign in the argument of $\mathbb{E}$. Observe that this construction is entirely independent of the nature of the set of constraints $\{\Phi_\alpha\}$.

Next, we continue to proceed formally in order to envisage how the projection operator method may be used in the case of gravity. Adopting the foregoing analysis, we suggest that

$$\begin{aligned}\mathbb{E}_P &= \mathbb{E}(\Sigma_{p=1}^P\, 2^{-p}\,[\,\Sigma_{A=1}^3\,(\mathcal{H}_{(p)\,A})^2 + \mathcal{H}_{(p)}^2\,]) \\ &= \int \mathbf{T}\, e^{-i\,\Sigma_{p=1}^P \int [\Sigma_{A=1}^3 N_{(p)\,A}\,\mathcal{H}_{(p)\,A} + N_{(p)}\,\mathcal{H}_{(p)}]\, dt}\, \mathcal{D}R(N^a, N)\end{aligned} \tag{61}$$

for an appropriately defined formal measure R. Observe that the integral over the measure R may well integrate over degrees of freedom that are not present in the time-ordered product (such as for $p > P$); however, there is no harm in doing so since R is already defined so that $\int \mathcal{D}R = 1$. We may also go one step further and assert that as $P \to \infty$ we obtain the formal expression

$$\mathbb{E} = \int \mathbf{T}\, e^{-i\int [N^a \mathcal{H}_a + N\mathcal{H}]\, d^3x\, dt}\, \mathcal{D}R(N^a, N)\ , \tag{62}$$

which, heuristically at least, realizes the projection operator that projects the original Hilbert space onto a correspondingly regularized physical Hilbert space.

C. Functional integral representation of the reproducing kernel for the physical Hilbert space

In Sec. II, we presented in (25) a continuous-time regularized functional integral representation of the reproducing kernel $\langle \pi'', g'' | \pi', g' \rangle$ for the original Hilbert space. The reproducing kernel for the (regularized) physical Hilbert space is given, in turn, by the expression $\langle \pi'', g'' | \mathbb{E} | \pi', g' \rangle$. In order to give this latter expression a functional integral representation we first regard

$$\int [N^a \mathcal{H}_a + N\mathcal{H}]\, d^3x \tag{63}$$

as a time-dependent "Hamiltonian" for some fictitious theory, in which case

$$\begin{aligned}
&\langle \pi'', g'' | \mathbf{T}\, e^{-i\int [N^a \mathcal{H}_a + N\mathcal{H}]\, d^3x\, dt} | \pi', g' \rangle \\
&\quad = \lim_{\nu\to\infty} \overline{\mathcal{N}}_\nu \int \exp\{-i\textstyle\int [g_{ab}\dot{\pi}^{ab} + N^a H_a + N H]\, d^3x\, dt\} \\
&\qquad \times \exp\{-(1/2\nu)\textstyle\int [b(x)^{-1} g_{ab} g_{cd} \dot{\pi}^{bc} \dot{\pi}^{da} + b(x) g^{ab} g^{cd} \dot{g}_{bc} \dot{g}_{da}]\, d^3x\, dt\} \\
&\qquad\quad \times \Pi_{x,t}\, \Pi_{a\le b}\, d\pi^{ab}(x,t)\, dg_{ab}(x,t)\ .
\end{aligned} \tag{64}$$

In this expression, there appear symbols $H_a(\pi, g)$ and $H(\pi, g)$ corresponding to the quantum operators $\mathcal{H}_a$ and $\mathcal{H}$. Superficially, these symbols may (formally) be identified with the classical diffeomorphism and Hamiltonian constraint functions, in which case the expression (64) contains in its phase,

and up to a surface term, the full Einstein action. Thus (64) comes ever closer to looking like a more traditional functional integral for gravity.

However, before integrating over the functions N^a and N and completing the story, we need to caution the reader that $\hbar$ is not zero (but rather one) and therefore the symbols H_a and H may not coincide with their usual classical expressions. All we can say at present is that H_a is a symbol for the operator $\mathcal{H}_a$, $a = 1, 2, 3$, and that H is a symbol for the operator $\mathcal{H}$. The connection between symbol and operator is implicitly contained in (64), and since, for the moment, the functions N_a and N are general functions within our control, we may use that fact to assert that

$$\begin{aligned}
&\langle\pi'', g''|\textstyle\int[M^a(y)\,\mathcal{H}_a(y) + M(y)\,\mathcal{H}(y)]\,d^3\!y\,|\pi', g'\rangle \\
&\quad = \lim_{\nu\to\infty} \overline{\mathcal{N}}_\nu \int e^{-i\int g_{ab}\,\dot{\pi}^{ab}\,d^3\!x\,dt} \textstyle\int[M^a(y)\,H_a(y,s) + M(y)\,H(y,s)]\,d^3\!y \\
&\qquad \times \exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da} + b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\,d^3\!x\,dt\} \\
&\qquad\quad \times \Pi_{x,t}\,\Pi_{a\le b}\,d\pi^{ab}(x,t)\,dg_{ab}(x,t)
\end{aligned} \tag{65}$$

for any smooth (test) functions M^a and M and for any time s, $0 < s < T$. Equation (65) gives an implicit connection between symbol and operator for the present theory. We observe that the more traditional connection between symbol and operator that normally holds for Wiener-regularized coherent-state path integrals [14] is unavailable in the present case since we are dealing with so-called weak coherent states for which no resolution of unity exists; see [1, 13].

In addition, thanks to analyticity in the present case, the diagonal matrix elements of an operator uniquely determine the operator, and so we can also assert the connection between symbol and operator (given by setting $\pi'', g'' = \pi', g'$),

$$\begin{aligned}
&\textstyle\int[M^a(y)\langle\pi', g'|\mathcal{H}_a(y)|\pi', g'\rangle + M(y)\langle\pi', g'|\mathcal{H}(y)|\pi', g'\rangle]\,d^3\!y \\
&\quad = \lim_{\nu\to\infty} \overline{\mathcal{N}}_\nu \int e^{-i\oint g_{ab}\,\dot{\pi}^{ab}\,d^3\!x\,dt} \textstyle\int[M^a(y)\,H_a(y,s) + M(y)\,H(y,s)]\,d^3\!y \\
&\qquad \times \exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da} + b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\,d^3\!x\,dt\} \\
&\qquad\quad \times \Pi_{x,t}\,\Pi_{a\le b}\,d\pi^{ab}(x,t)\,dg_{ab}(x,t)\ ,
\end{aligned} \tag{66}$$

again for smooth functions M^a and M and any s, $0 < s < T$. We note that $\langle\pi', g'|\mathcal{H}_a(y)|\pi', g'\rangle$ and $\langle\pi', g'|\mathcal{H}(y)|\pi', g'\rangle$ denote still other symbols that are

often associated with the local operators $\mathcal{H}_a(x)$ and $\mathcal{H}(x)$, respectively. In (66), the notation $\oint g_{ab}\,\dot{\pi}^{ab}\,d^3x\,dt$ means that only closed paths in phase space enter, i.e., just those paths for which the functions

$$\pi^{ab}(x,0) = \pi^{ab}(x,T) \equiv \pi'^{ab}(x)\ , \tag{67}$$

$$g_{ab}(x,0) = g_{ab}(x,T) \equiv g'_{ab}(x) \tag{68}$$

for all $x \in \mathcal{S}$. Note that a *closed* line integral in phase space involves just the symplectic form, and the result of the integral $\oint g_{ab}\dot{\pi}^{ab}\,d^3x\,dt$ is *invariant* under any (smooth) change of canonical coordinates.

Reproducing kernel for the physical Hilbert space

We now complete the story by interpreting the otherwise arbitrary c-number functions N^a and N as Lagrange multiplier functions and integrating them out of (64). Since N^a and N are not dynamical variables that must enter the formal phase-space functional integral measure in a prescribed way (i.e., as "$dp\,dq$"), we are free to integrate them as we choose—and we choose to integrate them in such a way as to *enforce the quantum constraints*, at least in a regulated fashion. As explained above, one natural way to achieve our goal involves the formal integration measure $\mathcal{D}R(N^a, N)$.

Combining several steps previously described, we now assert that the reproducing kernel for the regularized physical Hilbert space has the phase-space functional integral representation given by

$$\begin{aligned}
&\langle\pi'', g''|\mathbb{E}|\pi', g'\rangle \\
&\quad = \int \langle\pi'', g''|\mathbf{T}\,e^{-i\int[N^a\mathcal{H}_a + N\mathcal{H}]\,d^3x\,dt}|\pi', g'\rangle\,\mathcal{D}R(N^a, N) \\
&\quad = \lim_{\nu\to\infty}\overline{\mathcal{N}}_\nu \int e^{-i\int[g_{ab}\dot{\pi}^{ab} + N^a H_a + NH]\,d^3x\,dt} \\
&\qquad\quad \times \exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da} + b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\,d^3x\,dt\} \\
&\qquad\qquad \times \Big[\Pi_{x,t}\,\Pi_{a\le b}\,d\pi^{ab}(x,t)\,dg_{ab}(x,t)\Big]\,\mathcal{D}R(N^a, N)\ .
\end{aligned} \tag{69}$$

In this final expression we have reached our primary goal, at least from a formal perspective. Despite the general appearance of (69), we emphasize once again that this representation has been based on the affine commutation relations and *not* on any canonical commutation relations! Later, we

shall discuss a more careful definition of this formal expression along lines introduced in Sec. II, but for now let us examine (69) for its own sake.

We first comment on the range of integration for the "lapse" variable $N(x,t)$, a subject of recurrent interest [18]. Our view is that the range $-\infty < N(x,t) < \infty$ is the proper range when quantizing the theory. After all, in the Hamiltonian viewpoint, *space-time is a derived structure* of the classical theory. In principle, the issue here is no more complicated than for the reparametrized one-dimensional free particle. For this example, the original classical action is $I = \int[p\dot{q} - \frac{1}{2}p^2]\,dt$, where $\dot{q} \equiv dq/dt$, and has solutions $p(t) = p_o$ and $q(t) = p_o t + q_o$. The reparametrized version is given by $I' = \int[pq^* + st^* - \lambda(s + \frac{1}{2}p^2)]\,d\tau$, where $q^* \equiv dq/d\tau$, $t^* \equiv dt/d\tau$, and has solutions $p(\tau) = p_o$, $s(\tau) = s_o = -\frac{1}{2}p_o^2$, $t(\tau) \equiv \int_0^\tau \lambda(\sigma)\,d\sigma$, and $q(\tau) = t(\tau)p_o + q_o$. The function $\lambda(\tau)$ is essentially arbitrary. If, for example, $\lambda(\tau) = 3\tau^2 - 1$, i.e., $t(\tau) = \tau(\tau^2 - 1)$, the solution seems "to go backward in time", but that interpretation gives to the variable τ an unwarranted physical significance. The given solution is not wrong, it just repeats itself for a while. We can avoid a repeating behavior, e.g., by simply dropping the interval $-1 \leq \tau < 1$. No such issues occur if we require that $\lambda(\tau) > 0$ for all τ, in which case τ does indeed merit the name of "reparametrized time". By analogy, the function $N(x,t)$—which we have loosely called the lapse function—only deserves that name when, in the classical solution space, we insist that $N(x,t) > 0$; otherwise it is just another Lagrange multiplier function, no more and no less.

When one starts from a classical perspective, with its focus on physically relevant functions $N(x,t)$ which are strictly positive, it is a conceptual leap to change to functions $N(x,t)$ that can take on both signs [18]. However, when one starts from the quantum theory, as we have done, there is no such leap to make.

As a second topic regarding (69) we focus on its general structure. With H_a and H formally equal to the constraint functions of gravity (possibly up to terms in $\hbar$), the action appearing in the phase factor is indeed appropriate to gravity [18]. Moreover, the domain of integration is restricted to positive-definite metrics $\{g_{ab}(x,t)\} > 0$. Indeed, the particular ν-dependent regularizing phase-space metric in (69) *prevents the metric variable* $\{g_{ab}\}$ *from escaping the positive-definite domain.*

As a useful analogy, we note that the two-dimensional phase space metric

$$\beta^{-1}q^2\,dp^2+\beta\,q^{-2}\,dq^2 \tag{70}$$

is geodesically complete in the half-space $(p,q)\in\mathbb{R}\times\mathbb{R}^+$, and when it is part of a Brownian motion measure, as in the formal expression [cf., (38)]

$$\mathcal{N}\,e^{-(1/2\nu)\int[\beta^{-1}\,q^2\,\dot{p}^2+\beta\,q^{-2}\,\dot{q}^2]\,dt}\,\mathcal{D}p\,\mathcal{D}q\;, \tag{71}$$

it automatically restricts the Brownian motion trajectories to the half-space $\mathbb{R}\times\mathbb{R}^+$.

Lastly, we comment on the formal functional integration measure in (69), specifically for the Lagrange multiplier functions N^a and N. The formal measure $\mathcal{D}R$ has been defined earlier and is unlike conventional measures chosen for such variables. As emphasized here, and elsewhere [5], the measure $\mathcal{D}R$ is *designed to implement the quantum constraints*—as befits a quantum theory—and it has *not* been selected to enforce the classical constraints. Observe well that we have not "blindly postulated" the functional integral (69), but instead it has been *derived* as a specific functional integral representation of the well-chosen quantum matrix elements on the left-hand side. Unfamiliar as the measure $\mathcal{D}R$ may appear, we maintain that $\mathcal{D}R$—or some other measure equivalent to it—is the proper measure to choose to achieve our goal. Whether different treatments of the Lagrange multiplier functions that have been adopted by other workers are indeed equivalent or not to the use of $\mathcal{D}R$ is an interesting question, but it is not one we pursue here.

IV. DISCUSSION

In the preceding analysis, we have been strongly guided by the operator structure of the assumed theory of affine quantum gravity, and this discussion has led us to the formal functional integral representation (69) for the desired matrix elements. As was previously the case [cf., (29)], we now wish to turn (69) around and adopt the formal functional integral as our starting point and, in effect, use that expression to *define* the reproducing kernel for the regularized physical Hilbert space. Specifically, for that purpose, let us focus on the formal expression

$$\lim_{\nu\to\infty}\overline{\mathcal{N}}_\nu\int e^{i\int[\pi^{ab}\dot{g}_{ab}-N^aH_a-NH]\,d^3x\,dt}$$

$$\times \exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot\pi^{bc}\dot\pi^{da} + b(x)g^{ab}g^{cd}\dot g_{bc}\dot g_{da}]\,d^3x\,dt\}$$
$$\times\mathcal{D}\pi\,\mathcal{D}g\,\mathcal{D}R(N^a,N)\;. \tag{72}$$

Note the change of the kinematic term, which simply amounts to a phase factor in the definition of the coherent states. Also we have introduced the common shorthand $\mathcal{D}\pi\,\mathcal{D}g$ for the bracketed term in (69). For the sake of discussion, we shall refer to (72) as the "nonstandard expression". Our goal in this section is to discuss the nonstandard expression and see what steps are necessary to give it a proper meaning. We shall do so in a three step procedure: First, we compare the "standard" (see below) and "nonstandard" expressions. Second, we discuss a regularization and its removal that ultimately involves the elimination of the scalar density $b(x)$. Third, we examine the aspect of the problem that normally accounts for the perturbative nonrenormalizability of the gravitational field.

A. First look at the Nonstandard Expression

It is interesting to compare (72) with what we refer to as the standard expression for a phase-space functional integral for gravity. By the "standard expression" we mean the formal functional integral

$$\mathcal{M}\int e^{i\int[\pi^{ab}\dot g_{ab}-N^a\,H_a-N\,H]\,d^3x\,dt}\,\mathcal{D}\pi\,\mathcal{D}g\,\mathcal{D}N\;, \tag{73}$$

where $\mathcal{D}N\equiv\Pi_{x,t}\,dN(x,t)\,\Pi_a\,dN^a(x,t)$. In several important ways, the standard expression is *very* different than the nonstandard expression. Let us first comment on some of those differences. Much as it would be nice to think otherwise, one must recognize that the standard expression (73) is *totally undefined* as it stands; it is little more than a fancy way of writing $0\times(\infty)$. It begs for a definition as the limit of meaningful expressions [much as $0\times(\infty)$ may, for example, be defined as $\lim_{x\to 0}x\times(7/x)=7$], but what set of meaningful expressions should be chosen in the gravitational case is far from clear. A lattice limit? But then, what form should the regularized lattice expressions take? Symmetry and covariance offer only limited guidance. In point of fact, this same question faces any standard phase-space path integral, even that for a single degree of freedom in which a conventional lattice definition—as originally envisaged by Tobocman [19]—makes certain assumptions about the nature of the phase-space coordinates which

may or may not be true. [**Remark:** The skeptical reader is urged to propose a lattice prescription to quantize the nonrelativistic free particle of unit mass by a phase-space path integral whose Hamiltonian H is expressed in canonical coordinates p and q such that $H = \frac{1}{2}(p^2 + q^2)$.]

A further complication of the standard functional integral expression (73) arises from the *unbounded nature* of the formal integral $\int \cdots \mathcal{D}N$ over the Lagrange multiplier variables. This choice of integration measure, which is designed to enforce the classical constraints and thereby reduce the classical phase space before quantization, necessitates *gauge fixing* (to eliminate concomitant divergences) which reduces the classical phase space further to the physical phase space (at least locally) where each point labels a physically distinct state. Quantization on the reduced phase space is formally aided by the introduction of some additional factor (e.g., a Faddeev-Popov determinant, or its analogue), which may well lead to significant (Gribov) ambiguities which require a substantial modification of the functional integral [20, 7], and give rise to serious problems (such as unitary violation) within a BRST formulation [21].

Nonstandard expression

Let us raise similar issues regarding the nonstandard expression (72). Although (72) is formal as it stands, it can, to a considerable extent, already be regarded as "nearly" well defined. As noted in Sec. II, we can combine several factors together to make, for all finite ν, a positive, countably-additive, pinned measure on generalized functions. What makes (72) not well defined is the fact that the formal integrand does not constitute an integrable function with respect to that measure. We have already encountered that problem in Sec. II before any constraints were introduced, and we found that we could overcome that problem by regularizing the integrand and removing that regularization as a final step. Superficially, the same property holds when the constraint functions are present (say for fixed Lagrange multiplier values), save for one very important distinction (involving the field operator representation) which we shall address in the second point of discussion below.

Regarding the integration over the Lagrange multiplier variables, we emphasize the vast difference afforded by the projection operator method. First and foremost is the fact that *no gauge fixing* is introduced, no ghosts are

used, no Faddeev-Popov determinant (or its analogue) arises, and consequently, no Gribov ambiguities can exist. These properties arise, largely, because $\int \mathcal{D}R(N^a, N) = 1$, while $\int \mathcal{D}N = \infty$. The difference here could not be greater, and it arises because in the former case one quantizes first and reduces second, while in the latter case one reduces first and quantizes second. Except in basically trivial cases, the second option is fraught with substantial obstacles [7].

One of the most significant differences between the standard and the nonstandard expressions refers to the representation of the quantum mechanical amplitudes that is involved. For the standard expression, it is usually assumed that (73) leads to a representation in which the metric field operator $\hat{g}_{ab}(x)$ is diagonalized and thus is sharply represented. In combination with any needed auxiliary factor in the functional integral, diffeomorphism invariance suggests that (73) depends only on the "geometry" of the initial and final three-surfaces, and not on the details of any specific metric expressions [16, 18]. An analogous view is also prominent in the associated "loop quantum gravity" in which, e.g., bras and kets depend only on knot invariants as labels of "physically" distinguishable states [22]. In contrast, the representation afforded by the nonstandard expression (72) is that of a *coherent-state representation*, which depends on smooth metric g_{ab} and momentum π^{ab} fields, that represent not *sharp* operator (eigen)values but *mean* operator values, e.g.,

$$\langle \pi, g|\, \hat{\pi}^{ab}(x)\, |\pi, g\rangle = \pi^{ab}(x)\ , \tag{74}$$

$$\langle \pi, g|\, \hat{g}_{ab}(x)\, |\pi, g\rangle = g_{ab}(x)\ , \tag{75}$$

$$\langle \pi, g|\, \hat{\pi}_a^b(x)\, |\pi, g\rangle = g_{ac}(x)\, \pi^{cb}(x)\ . \tag{76}$$

Note well, that besides the local self-adjoint metric $\hat{g}_{ab}(x)$ and scale field $\hat{\pi}_a^b(x)$, we have used the local *symmetric* (but *not* self-adjoint) momentum operator $\hat{\pi}^{ab}(x)$ in the first of these expressions. These expectations are not gauge invariant, nor should they be, since they are taken in the "original" Hilbert space where the constraints are not fulfilled. The gauge invariant part of the metric field, for example, and in so far as the regularized physical Hilbert space is concerned, is determined by the matrix elements

$$\langle \pi'', g''|\, \mathbb{E}\, g_{ab}(x)\, \mathbb{E}\, |\pi', g'\rangle\ , \tag{77}$$

which is an expression that does not require restricting the functional dependence of the bras and kets.

B. Second Look at the Nonstandard Expression

In interpreting (72) we have concluded above that we must first regularize the integrand in order to obtain an integrable function. For the kinematic term $\int \pi^{ab}\,\dot{g}_{ab}\,d^3x\,dt$—and even for the diffeomorphism constraint contribution $-\int N^a H_a\,d^3x\,dt$—any natural regularization, such as one based on the expansion functions $\{h_p(x)\}$ and $\{f^a_{p\,A}(x)\}$, or a lattice formulation as discussed in Sec. II, will be compatible regularizations. For these terms alone, the limit of the regularized functional integral as the regularization is removed will converge to the desired result. The implication of this fact is that these parts of the integrand are compatible with the initial (ultralocal) representation of the field operators; in fact, they are compatible with any diffeomorphism invariant realization. However, when it comes time to consider the Hamiltonian constraint, the behavior is quite different. While it is true that regularizing the Hamiltonian constraint will lead to a set of well-defined functional integrals, the limit of such regularized expressions will *not* converge to an acceptable result. There are two basic and important reasons for this unsatisfactory behavior, one of which (wrong field operator representation) we will deal with in this section, the other of which (perturbative nonrenormalizability of gravity) we will discuss in the next section.

The first reason for the lack of a suitable convergence of the regularized nonstandard functional integral relates to the fact that the representation of the field operators needed to satisfy the Hamiltonian constraint is unitarily *in*equivalent to the ultralocal representation imposed in the initial stage of the quantization procedure. It is at the present stage of the analysis that we finally encounter the fact that our initial choice of field operator representation is incompatible with making the Hamiltonian constraint operator $\mathcal{H}(x)$ into a densely defined local operator. Stated otherwise, using the ultralocal operator representation, the operator $\int N(x,t)\,\mathcal{H}(x)\,d^3x\,dt$, for any nonzero smooth function $N(x,t)$, has only the zero vector in its domain. This defect must be fixed before proceeding, and in so doing we will be explicitly led to a new representation of the field operators, one that is unitarily inequivalent to our starting (ultralocal) representation. In the process of effecting this change of representation, the scalar density $b(x)$ will disappear from the scene entirely!

Pedagogical example

It is pedagogically useful to outline an analogous story for a simpler and more familiar example. Consider general, locally self-adjoint field and momentum operators, $\hat{\phi}(x)$ and $\hat{\pi}(x)$, $x \in \mathbb{R}^3$, which satisfy the canonical commutation relations

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x-y) \; . \tag{78}$$

Build a set of coherent states

$$|\pi, \phi\rangle \equiv e^{i[\hat{\phi}(\pi)-\hat{\pi}(\phi)]}\,|\eta\rangle \; , \tag{79}$$

where $\hat{\phi}(\pi) \equiv \int \hat{\phi}(x)\pi(x)\,d^3\!x$ and $\hat{\pi}(\phi) \equiv \int \hat{\pi}(x)\phi(x)\,d^3\!x$, with π and ϕ real test functions, and $|\eta\rangle$ is a normalized but otherwise unspecified fiducial vector. Note well that the choice of $|\eta\rangle$ in effect determines the representation of the canonical field operators. We next present a portion of the story from Ref. 8.

We initially choose $|\eta\rangle$ to correspond to an ultralocal representation such that

$$\begin{aligned} \langle\pi'', \phi''|\pi', \phi'\rangle &= \exp\{\tfrac{1}{2}i\textstyle\int[\phi''(x)\pi'(x) - \pi''(x)\phi'(x)]\,d^3\!x\} \\ &\times \exp\boldsymbol{(}-\tfrac{1}{4}\textstyle\int\{M(x)^{-1}[\pi''(x)-\pi'(x)]^2 + M(x)[\phi''(x)-\phi'(x)]^2\}\,d^3\!x\,\boldsymbol{)} \; . \end{aligned} \tag{80}$$

Here $M(x)$, $0 < M(x) < \infty$, is an arbitrary (smooth) function of the ultralocal representation with the dimensions of M. The given ultralocal field operator representation is in fact unitarily inequivalent for each distinct function $M(x)$. [Note well that $M(x)$ here plays the role of $b(x)$ in the present paper.]

We wish to apply this formulation to describe the *relativistic free field of mass m* for which the Hamiltonian operator is formally given by

$$\mathcal{H} = \tfrac{1}{2}\textstyle\int :\{\hat{\pi}(x)^2 + [\nabla\hat{\phi}(x)]^2 + m^2\hat{\phi}(x)^2\}:\,d^3\!x \; . \tag{81}$$

If we build this operator out of the field and momentum operators in the ultralocal representation, then no matter what vector is used to define : :, $\mathcal{H}$ will have only the zero vector in its domain. We need to change the field operator representation, which means we have to change the fiducial

vector from $|\eta\rangle$ to $|0;m\rangle$, the true ground state of the proposed Hamiltonian operator $\mathcal{H}$.

Let us first regularize the formal Hamiltonian $\mathcal{H}$. To that end, let $\{u_n(x)\}$ denote a complete set of real, orthonormal functions on $\mathbb{R}^3$ and define the sequence of kernels, for all $N \in \{1,2,3,\ldots\}$, given by

$$K_N(x,y) \equiv \sum_{n=1}^{N} u_n(x)\, u_n(y)\ , \tag{82}$$

which converges to $\delta(x-y)$ as a distribution when $N \to \infty$. Let

$$\hat{\phi}_N(x) \equiv \textstyle\int K_N(x,y)\,\hat{\phi}(y)\,d^3\!y\ , \tag{83}$$

$$\hat{\pi}_N(x) \equiv \textstyle\int K_N(x,y)\,\hat{\pi}(y)\,d^3\!y\ , \tag{84}$$

and with these operators build the sequence of regularized Hamiltonian operators

$$\mathcal{H}_N \equiv \tfrac{1}{2}\textstyle\int\, :\{\hat{\pi}_N(x)^2 + [\nabla\hat{\phi}_N(x)]^2 + m^2\hat{\phi}_N(x)^2\}:\ d^3\!x \tag{85}$$

for all N, where $:\ :$ denotes normal order with respect to the ground state $|0;m\rangle_N$ of $\mathcal{H}_N$.

We would like to have a constructive way to identify the ground state of $\mathcal{H}_N$. For this purpose consider the set

$$S_N \equiv \left\{ \frac{\Sigma_{j,k=1}^{J}\, a_j^*\, a_k\,\langle\pi_j,\phi_j|\,e^{-\mathcal{H}_N^2}\,|\pi_k,\phi_k\rangle}{\Sigma_{j,k=1}^{J}\, a_j^*\, a_k\,\langle\pi_j,\phi_j|\pi_k,\phi_k\rangle}\ :\ J<\infty \right\} \tag{86}$$

for general sets $\{a_j\}$ (not all zero), $\{\pi_j\}$, and $\{\phi_j\}$. (How these expressions may be generated is discussed in Ref. 8.) As N grows, the general element in S_N becomes exponentially small, save for elements that correspond to vectors which well approximate the ground state $|0;m\rangle_N$. Suitable linear combinations can convert the original reproducing kernel $\langle\pi'',\phi''|\pi',\phi'\rangle$ to the reproducing kernel $\langle\pi'',\phi'';m|\pi',\phi';m\rangle_N$ which is based on a fiducial vector that has the form $|0;m\rangle_N$ for the first N degrees of freedom and is unchanged for the remaining degrees of freedom. Finally, we may take the limit $N\to\infty$ which then leads to

$$\langle\pi'',\phi'';m|\pi',\phi';m\rangle = \exp\{\tfrac{1}{2}i\textstyle\int[\tilde{\phi}''^*(k)\,\tilde{\pi}'(k) - \tilde{\pi}''^*(k)\,\tilde{\phi}'(k)]\,d^3\!k\}$$
$$\times\exp\big(-\tfrac{1}{4}\textstyle\int\{\omega(k)^{-1}|\tilde{\pi}''(k)-\tilde{\pi}'(k)|^2 + \omega(k)|\tilde{\phi}''(k)-\tilde{\phi}'(k)|^2\}\,d^3\!k\big)\ , \tag{87}$$

where $\omega(k) \equiv \sqrt{k^2+m^2}$ and $\tilde{\pi}(k) \equiv (2\pi)^{-3/2}\int e^{-ik\cdot x}\,\pi(x)\,d^3\!x$, etc. The procedure sketched above is referred to as *recentering the coherent states* or equivalently as *recentering the reproducing kernel.* This form of reproducing kernel is no longer ultralocal and contains no trace of the scalar function $M(x)$, whatever form it may have had. Moreover, and this is an important point, the new representation is fully compatible with the Hamiltonian $\mathcal{H}$ being a nonnegative, self-adjoint operator. Indeed, the expression for the propagator is given by

$$\langle\pi'',\phi'';m|e^{-i\mathcal{H}T}|\pi',\phi';m\rangle = L''\,L'\,\exp[\textstyle\int\tilde{\zeta}''^*(k)\,e^{-i\omega(k)T}\,\tilde{\zeta}'(k)\,d^3\!k]\;, \tag{88}$$

where

$$\tilde{\zeta}(k) \equiv [\omega(k)^{1/2}\,\tilde{\phi}(k) + i\omega(k)^{-1/2}\,\tilde{\pi}(k)]/\sqrt{2}\;, \tag{89}$$

$$L \equiv \exp[-\tfrac{1}{2}\textstyle\int|\tilde{\zeta}(k)|^2\,d^3\!k]\;. \tag{90}$$

The definition offered by (88) is continuous in T, which is the principal guarantor that the expression

$$\mathcal{H} = \tfrac{1}{2}\textstyle\int :\{\hat{\pi}(x)^2 + [\nabla\hat{\phi}(x)]^2 + m^2\,\hat{\phi}(x)^2\}:\,d^3\!x\;, \tag{91}$$

where : : denotes normal ordering with respect to the ground state $|0;m\rangle$ of the operator $\mathcal{H}$, is a self-adjoint operator as desired.

Let us summarize the basic content of the present pedagogical example. Even though we started with a very general ultralocal representation, as characterized by the general function $M(x)$, we have forced a complete change of representation to one compatible with the Hamiltonian operator for a relativistic free field of arbitrary mass m. In so doing all trace of the initial arbitrary function $M(x)$ has disappeared, and in its place, effectively speaking, has appeared the pseudo-differential operator $\sqrt{-\nabla^2+m^2}$ having only its dimension (mass) in common with the original function $M(x)$. The original ultralocal representation is completely gone! **(Remark:** A moments reflection should convince the reader that a comparable analysis can be made for either the interacting ϕ^4_2 or ϕ^4_3 model as well, both of which satisfy (78), showing that the general argument is not limited just to free theories; see [8].**)**

Strong coupling gravity

The discussion in the present paper has been predicated on the assumption that we are analyzing the gravitational field and therefore the classical Hamiltonian is that given in (43). However, it is pedagogically instructive if we briefly comment on an approximate theory—based on the so-called "strong coupling approximation" [23]—where the Hamiltonian constraint (43) is replaced by the expression

$$H_{SCA}(x) \equiv g(x)^{-1/2}[\,\pi^a_b(x)\,\pi^b_a(x) - \tfrac{1}{2}\pi^a_a(x)\,\pi^b_b(x)\,] + 2\Lambda\,g(x)^{1/2}\,, \tag{92}$$

in which the term proportional to ${}^{(3)}R(x)$ has been dropped, and where we have also introduced the cosmological constant Λ (with dimensions L^{-2}) and an associated auxiliary term in the Hamiltonian. The result is a model for which the new Hamiltonian constraint (92) is indeed compatible with some form of an ultralocal representation. The proper form of that ultralocal representation may be determined by a similar procedure, i.e., by studying an analogue of the set S_N, and by recentering the reproducing kernel based on ensuring that the quantum version of $H_{SCA}(x)$ is a local self-adjoint operator. In so doing, we note that it may happen that not all arbitrariness of the original scalar density $b(x)$ is "squeezed out" by the recentering procedure described above. This situation occurs because a one-parameter arbitrariness generally remains for typical ultralocal models [12]. (Any remaining arbitrariness in the ultralocal case is in contrast with that of the true Hamiltonian constraint for gravity for which we expect no trace of the original function $b(x)$ to remain.) These remarks conclude our discussion of strong coupling gravity.

C. Third Look at the Nonstandard Expression

The preceding discussion has been based on the assumption that some fiducial vector can be found compatible with the Hamiltonian operator constraint, or stated otherwise, that the Hamiltonian constraint can actually be realized as a local self-adjoint operator. This requirement is by no means obvious, and it is to this issue that we now turn our attention. The difficulty arises because the naive form of the Hamiltonian constraint operator almost surely needs some form of renormalization if it is going to be well defined. If perturbation theory is any guide, we not only expect that there will be renormalization

counterterms, but because gravity is perturbatively nonrenormalizable, one may expect an infinite number of distinct counterterms. On the other hand, as we next argue, it is possible that perturbation theory is not a very reliable guide in the case of perturbatively nonrenormalizable theories.

Nonrenormalizable scalar fields

Consider the case of perturbatively nonrenormalizable quartic, self-interacting scalar fields, i.e., the so-called ϕ^4_n theories, where the space-time dimension $n \geq 5$. On the one hand, viewed perturbatively, such theories entail an infinite number of distinct counterterms. On the other hand, the continuum limit of a straightforward Euclidean lattice formulation leads to a quasifree theory—a genuinely *noninteracting* theory—whatever choice is made for the renormalized field strength, mass, and coupling constant [24]. In the author's view both of these results are unsatisfactory. Instead, it is possible that an *intermediate behavior* holds true, even though that cannot be proven yet. Let us illustrate an analogous but simpler situation where the conjectured intermediate behavior can be rigorously established.

Consider an ultralocal quartic interacting scalar field, which, viewed classically, is nothing but the relativistic ϕ^4_n model with all the spatial gradients in the usual free term dropped. As a mathematical model of quantum field theory, an ultralocal model is readily seen to be perturbatively nonrenormalizable, while the continuum limit of a straightforward lattice formulation becomes quasifree, basically because of the vise grip of the Central Limit Theorem. Perturbative nonrenormalizability and lattice-limit triviality is similar to the behavior for relativistic ϕ^4_n models, but for the simpler ultralocal model for which an intermediate approach can be rigorously proven to hold [12]. Roughly speaking, a characterization of this intermediate behavior is the following: From a functional integral standpoint, and for any positive value of the quartic coupling constant, the quartic interaction acts like a *hard-core* in history space projecting out certain contributions that would otherwise be allowed by the free theory alone. This phenomenon takes the form of a nonstandard, nonclassical counterterm in the Hamiltonian that does *not* vanish as the coupling constant of the quartic interaction vanishes. Specifically, for the model in question, the additional counterterm is *a counterterm for the kinetic energy* and is formally proportional to $\hbar^2/\phi(x)^2$, which in form is not unlike the centripetal potential that arises in spherical coordinates in

three-dimensional quantum mechanics. In summary, inclusion of a formal additional interaction proportional to $\hbar^2/\phi(x)^2$ in the Hamiltonian density is sufficient to result in a well defined and nontrivial (i.e., non-Gaussian) quantum theory for interacting ultralocal scalar models. In addition, it may be shown [12] that the classical limit of such quantum theories reproduces the classical model with which one started.

The foregoing brief summary holds rigorously for the ultralocal scalar fields, and it is conjectured that a suitable counterterm would lead to an acceptable intermediate behavior for the relativistic models ϕ^4_n, $n \geq 5$. What form should the counterterm take in the case of the relativistic ϕ^4_n models? We can make a plausible suggestion guided by the following general principle that holds in the ultralocal case: The counterterm should be an ultralocal (because the kinetic energy is ultralocal) potential term arising from the kinetic energy. For the relativistic field that argument suggests the counterterm should again be proportional to $\hbar^2/\phi(x)^2$. It is also part of this general conjecture that the same counterterm is not limited to ϕ^4_n models, but should be effective for other nonrenormalizable interactions, e.g., such as ϕ^6_n, $n \geq 4$, etc. The full argument available at present to support this conjecture appears in Chap. 11 of [12]. (It may even be possible to exam this proposal by means of suitable Monte Carlo studies, but so far this challenge has not been taken up.)

Note well that the hard-core picture of nonrenormalizable interactions leads to such interactions behaving as *discontinuous perturbations*: Once turned on, such interactions cannot be completely turned off! Stated otherwise, as the nonlinear coupling constant is reduced toward zero, the theory passes continuously to a "pseudofree" theory—different than the "free" theory—*which retains the effects of the hard core*. The interacting theory is *continuously connected* to the pseudofree theory, and may even possess some form of perturbation theory about the pseudofree theory. Evidently, the presence of the hard-core interaction makes any perturbation theory developed about the original unperturbed theory almost totally meaningless.

Nonrenormalizable gravity

Although the differences between gravity and nonrenormalizable scalar interaction are significant in their details, there are certain similarities we wish to draw on. Most importantly, one can argue [25] that the nonlinear contribu-

tions to gravity act as a hard-core interaction in a quantization scheme, and thus the general picture sketched above for nonrenormalizable scalar fields should apply to gravity as well. Assuming that the analogy holds further, there should be a nonstandard, nonclassical counterterm that incorporates the dominant, irremovable effects of the hard-core interaction. Accepting the principle that in such cases perturbation theory offers no clear hint as to what counterterms should be chosen, we appeal to the guide used in the scalar case. Thus, as our proposed counterterm, we look for an ultralocal potential arising from the kinetic energy that appears in the Hamiltonian constraint. In fact, the only ultralocal potential that has the right transformation properties is proportional to $\hbar^2 g(x)^{1/2}$. Thus we are led to conjecture that the "nonstandard counterterm" is none other than a term like the familiar cosmological constant contribution! Unlike the scalar field which required an unusual term proportional to $1/\phi(x)^2$, the gravitational case has resulted in suggesting a term proportional to an "old friend", namely $g(x)^{1/2}$. At first glance, it seems absurd that such a harmless looking term could act to "save" the nonrenormalizability of gravity. In its favor we simply note that the analogy with how other nonrenormalizable theories are "rescued" is too strong to dismiss the present proposal out of hand—and of course one must resist any temptation "to think perturbatively". Any attempt to consider this possibility must wait until another occasion; we hope to return to this subject elsewhere.

As one small aspect of this problem, let us briefly discuss how the factor $\hbar^2$ arises in the gravitational case. Merely from a *dimensional* point of view, we note that (the first term of) the local kinetic energy operator has a formal structure given by

$$-\frac{16\pi G}{c^3}\hbar^2\Big(\frac{\delta}{\delta g_{cb}(x)}\,g_{ac}(x)\,g(x)^{-1/2}\,g_{bd}(x)\,\frac{\delta}{\delta g_{da}(x)}-\cdots\Big)\;, \tag{93}$$

where we have restored the factor $16\pi G/c^3$. Thus the anticipated counterterm is proportional to $(G\hbar^2/c^3)\,g(x)^{1/2}$. We next cast this term into the usual form for a contribution to the potential part of the Hamiltonian constraint, namely, in a form proportional to $(c^3/G)\,\Lambda\,g(x)^{1/2}$. Hence, to recast our anticipated counterterm into this form, we need a factor proportional to

$$\frac{G^2\hbar^2}{c^6}\equiv l^4_{Planck}\approx(10^{-33}cm)^4\;. \tag{94}$$

In the classical symbol for the Hamiltonian constraint operator, this factor is multiplied by an expectation value with dimensions L^{-6} originating from the density nature of the two momentum factors and leading to an overall factor with the dimensions L^{-2} that is proportional to $\hbar^2$ as claimed. Let us call the resultant counterterm $\Lambda_C\, g(x)^{1/2}$. Since the sign of the DeWitt metric that governs the kinetic energy term is indefinite, it is not even possible to predict the sign of Λ_C. However, one thing appears certain. While the proposed counterterm $\Lambda_C\, g(x)^{1/2}$ is certainly not cosmological in origin, its influence may well be!

The foregoing scenario has assumed the hard-core model of nonrenormalizable interactions applies to the theory of gravity. However, that may well not be the case, and, instead, some other counterterm(s) may be required to cure the theory of gravity. Note well that the general structure of our approach to quantize gravity is largely insensitive to just what form of regularization and renormalization is required. In particular, the use of the affine field variables, the application of the projection operator method to impose constraints, and the development of the nonstandard phase-space functional integral representation for the reproducing kernel of the regularized physical Hilbert space all have validity quite independently of the form in which the Hamiltonian constraint is ultimately turned into a local self-adjoint operator. Although we have outlined one particular version in which the Hamiltonian constraint may possibly be made into a densely defined local operator, we are happy to keep an open mind about the procedure by which this ultimately may take place since many different ways in which this process can occur are fully compatible with the general principles of our proposed quantization scheme for the gravitational field.

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